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Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes

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Abstract: In this paper, the connection between the Lorentz-covariant counterterms that regularize the four-dimensional AdS gravity action and topological invariants is explored. It is shown that demanding the spacetime to have a negative constant curvature in the asymptotic region permits the explicit construction of such series of boundary terms.
The orthonormal frame is adapted to appropriately describe the boundary geometry and, as a result, the boundary term can be expressed as a functional of the boundary metric, extrinsic curvature and intrinsic curvature. This choice also allows to write down the background-independent Noether charges associated to asymptotic symmetries in standard tensorial formalism.
The absence of the Gibbons-Hawking term is a consequence of an action principle based on a boundary condition different than Dirichlet on the metric. This argument makes plausible the idea of regarding this approach as an alternative regularization scheme for AdS gravity in all even dimensions, different than the standard counterterms prescription.
As an illustration of the finiteness of the charges and the euclidean action in this framework, the conserved quantities and black hole entropy for four-dimensional Kerr-AdS are computed.

Keywords: Black Holes, Classical Theories of Gravity.

Contents

1. Introduction

The construction of background-independent conserved quantities in AdS gravity has attracted the attention of several authors in the recent literature, especially in the context of AdS/CFT correspondence [1, 2].

In the ref. [3] we consider the addition of the Euler term to the four-dimensional AdS gravity action, that leads to a background-independent charge definition by means of the Noether theorem for asymptotic symmetries. For that case, the total action can be expressed in the language of differential forms as

$$I_4 = \frac{l^2}{64\pi} \int_M \hat{\epsilon}_{ABCD} \left(\hat{R}^{AB} + \frac{1}{l^2} e^A e^B \right) \left(\hat{R}^{CD} + \frac{1}{l^2} e^C e^D \right) \tag{1.1}$$

where $e^A = e^A_\mu dx^\mu$ is the vierbein (local orthonormal frame) and $\hat{R}^{AB} = \frac{1}{2} \hat{R}^{AB}_{\mu\nu} dx^\mu \wedge dx^\nu$ is the 2-form Lorentz curvature constructed up from the spin connection $\omega^{AB} = \omega^{AB}_\mu dx^\mu$ as $\hat{R}^{AB} = d\omega^{AB} + \omega^A_C \omega^{CB}$. The symbol $\hat{\epsilon}_{ABCD}$ is the totally antisymmetric Levi-Civita tensor and the Latin indices run in the set $A = \{0, 1, 2, 3\}$. The hatted curvatures stand for $4-$ dimensional ones, the wedge product $\wedge$ between the differential forms is understood and l is the AdS radius.

The action (1.1) does not require the addition of any boundary term to cancel the divergences that appear in the evaluation of the euclidean continuation (see the discussion below). As the Euler term is quadratic in the Riemann curvature in four dimensions, it coincides with the Gauss-Bonnet term, so that the action (1.1) in standard tensorial notation is

$$I_4 = -\frac{1}{16\pi}\int_M d^4x\sqrt{-\hat{g}}\left(\hat{R} - 2\Lambda + \frac{l^2}{4}(\hat{R}^{\mu\nu\sigma\rho}\hat{R}_{\mu\nu\sigma\rho} - 4\hat{R}^{\mu\nu}\hat{R}_{\mu\nu} + \hat{R}^2)\right) \tag{1.2}$$

with the cosmological constant $\Lambda = -3/l^2$. The coupling constant in front of the Euler-Gauss-Bonnet term has been fixed demanding a novel condition on the asymptotic curvature rather than the standard Dirichlet condition on the metric. The crucial step is to assume that the spacetime has a constant (negative) curvature at the boundary, that is,

$$\hat{R}^{\mu\nu}_{\alpha\beta} = -\frac{1}{l^2}\delta^{[\mu\nu]}_{[\alpha\beta]}\,. \tag{1.3}$$

on ∂M. The above relation represents an asymptotic local condition but does not impose any further restriction on the global topology of the solution. For instance, eq. (1.3) is satisfied not only by point-like configurations (black holes) but also by extended objects (black strings, domain walls, etc.). The general character of this approach is suitable to treat a wide family of solutions, from topological black holes to Kerr-AdS and Taub-NUT/Bolt-AdS, with a single formula for the conserved quantities.

The presence of the Euler term in the action does not modify the equations of motion, but radically change the form of the conserved charges, canceling the typical divergences in spacetimes with cosmological constant. When evaluated for asymptotic Killing vectors, this formula renders a finite value and recovers the correct results for a large variety of solutions [3]. In particular, the charge formula obtained from this procedure corrects the anomalous factor in the Komar's potential [4, 5] in a background-independent framework.

For higher even dimensions ($D = 2n$), it was shown in ref. [6] that the addition of the Euler term (topological invariant constructed up with n Lorentz curvatures) always regularize the definition of conserved quantities in asymptotically AdS spacetimes.

Unfortunately, in odd-dimensional AdS gravity there is a severe obstruction to a similar construction because topological invariants of the Euler class do not exist for $D = 2n + 1$. Therefore, one can only supplement the action with boundary terms whose explicit form depends on the kind of boundary condition under consideration.

In the context of the AdS/CFT correspondence, the regularization of the AdS action using counterterms was carried out by Henningson and Skenderis [7, 8]. In this work, they presented a systematic procedure to reconstruct asymptotically AdS spacetimes for a given data of the boundary metric. The same algorithm obtains the explicit form of the counterterms required by the finiteness of the stress tensor. The conserved quantities defined through the quasilocal stress tensor [9] for this regularized action are background-independent and have been computed for a number of solutions. In particular, in five dimensions, the mass for Schwarzschild-AdS black holes appears to be shifted in a constant respect the hamiltonian one. This constant is interpreted as the Casimir energy of the corresponding boundary CFT [10]. A formula for the vacuum energy for any odd dimension $D = 2n+1$ was proposed by Emparan, Johnson and Myers [11] as an extrapolation from the results computed up to seven dimensions. This is essentially due to the technical difficulty to obtain the explicit form of the boundary terms in high enough D, what makes the full series of counterterms for any dimension still unknown.

In an alternative approach to deal with the above problem, a finite action principle for odd-dimensional AdS gravity was achieved supplementing the Einstein-Hilbert action by appropriate Lorentz-covariant boundary terms [12]. Apart from the condition on the asymptotic curvature (1.3), we demand a *holographic* condition on the extrinsic curvature of the boundary in order to make the on-shell action stationary. This procedure singles out the form of the boundary term for a given dimension as a functional of the boundary tensorial objects. The mass for static black holes calculated through the Noether theorem also introduces a vacuum energy for AdS spacetime, and explicitly verifies the expression conjectured in [11] for all odd dimensions.

In this paper, we go back to the even-dimensional case and explicitly construct the boundary terms that regularize the AdS action following a strategy similar to the one implemented in [12]. In the refs. [3, 6], the procedure that leads to the conserved quantities was carried out in first order formalism, in terms of the tetrad and the spin connection. Therefore, the final form of the charges has an explicit dependence on both fields, what introduces an ambiguity in the formula due to the arbitrariness in the choice of the orthonormal frame. This fact also makes difficult the comparison to other methods to compute conserved quantities in AdS gravity. Here, we show that a suitable choice of the tetrad removes such ambiguity and allow us to write down a tensorial expression for the charge in terms of the boundary metric, the boundary Riemann tensor and the extrinsic curvature.

2. Lorentz-covariant counterterms as surface terms in $D = 4$

In this section, we are interested in constructing an appropriate Lorentz-covariant boundary term for four-dimensional AdS gravity, so that the action has an extremum for arbitrary variations of the fields.

As we shall see below, this can be done integrating the surface term from the variation of the action once proper boundary conditions are imposed.

Let us consider the standard Einstein-Hilbert action with negative cosmological constant supplemented in a boundary term B_3. In the language of the tetrad and the spin connection this is written as

$$I_G = \frac{1}{32\pi} \int_M \hat{\epsilon}_{ABCD} \left(\hat{R}^{AB} e^C e^D + \frac{1}{2l^2} e^A e^B e^C e^D \right) + \int_{\partial M} B_3 \,. \tag{2.1}$$

An arbitrary variation of fields e^A and ω^{AB} produces the equations of motion for General Relativity and a surface term

$$\delta I_G = \int_M \varepsilon_A \delta e^A + \varepsilon_{AB} \delta \omega^{AB} + d\Theta \,, \tag{2.2}$$

where ε_A is the Einstein equation,

$$\varepsilon_A = \hat{\epsilon}_{ABCD}\left(\hat{R}^{BC} + \frac{1}{l^2}e^B e^C\right)e^D \tag{2.3}$$

and the equation $\varepsilon_{AB} = 0$ simply implies that the torsion must vanish since the tetrad is invertible.

In order to obtain the field equations we need to perform an integration by parts that gives the first contribution to the surface term Θ

$$\Theta = \frac{1}{32\pi}\hat{\epsilon}_{ABCD}\delta\omega^{AB}e^C e^D + \delta B_3 \tag{2.4}$$

where the second one is coming from the variation of the boundary term in eq. (2.1).

2.1 Adapted coordinates

We consider a radial foliation of the spacetime, where the line element is written in gaussian (normal) coordinates

$$ds^2 = \hat{g}_{\mu\nu}dx^\mu dx^\nu = N^2(r)dr^2 + h_{ij}(r,x)dx^i dx^j \tag{2.5}$$

that are useful to describe the boundary geometry. The boundary is located at a fixed value of $r = r_0$. The most natural choice of the local orthonormal frame is adapting the tetrad to the boundary, taking the block-diagonal decomposition

$$e^1 = N dr \tag{2.6}$$

$$e^a = e^a_i dx^i\,. \tag{2.7}$$

where we have separated the tangent space indices as $A = \{1, a\}$ and the spacetime ones as $\mu = \{r, i\}$. The indices of the tangent space and the spacetime are lowered and raised with η_{AB} and $\hat{g}_{\mu\nu}$, respectively. This preferred frame choice still preserves the rotational Lorentz invariance on the boundary as a residual symmetry of the fields.

As torsion vanishes, the spin connection can be completely determined in terms of the tetrad $\omega^{AB} = \omega^{AB}(e^A)$,

$$\omega^{AB}_\mu = -e^{B\nu}\nabla_\mu e^A_\nu \tag{2.8}$$

where ∇_μ is the covariant derivative in the Christoffel symbol. From the above relation, we can calculate the components ω^{1a}, that turn out to be related to the vielbein on the boundary by

$$\omega^{1a} = -K^j_i e^a_j dx^i = -K^a \tag{2.9}$$

where K_{ij} is the extrinsic curvature that in normal coordinates (2.5) is given by

$$K_{ij} = -\frac{1}{2N}h'_{ij}\,. \tag{2.10}$$

Here, the prime stands for the derivative in the radial coordinate. However, for the vielbein choice (2.6), (2.7), the components of the Lorentz connection ω^{ab} are not expressed in terms of tensorial quantities on the boundary. That is a problem if we are interested in establishing a connection to the usual tensorial formalism, that is, the boundary term B_3 to be expressible as a local function of the boundary metric h_{ij}, the extrinsic curvature K_{ij} and the intrinsic curvature $R^{kl}_{ij}(h)$ (Riemann tensor of the boundary metric).

Then, the aim is constructing up Lorentz-covariant boundary terms with the ingredients we have in the formalism of the spin connection and vierbein. However, we cannot use directly the spin connection, because it is not a vector for Lorentz transformations. In order to restore the Lorentz covariance, we define the second fundamental form (SFF) as the difference of two spin connections at the boundary,

$$\theta^{AB} = \omega^{AB} - \bar{\omega}^{AB} \,, \tag{2.11}$$

where ω^{AB} is the dynamical field (the one that is varied to obtain the corresponding field equation) and $\bar{\omega}^{AB}$ is a fixed reference that lives only on the boundary. We are assuming that $\bar{\omega}^{AB}$ transforms in the same way as ω^{AB} under the action of the group SO(3,1), but not under functional variations that act only on the dynamical fields.

In the vicinity of the boundary ($r = r_0$), we can always write down a product metric

$$ds^2 = N^2(r)dr^2 + \bar{h}_{ij}(x)dx^i dx^j \tag{2.12}$$

such that the matching condition is given by $\bar{h}_{ij}(x) = h_{ij}(r = r_0, x)$ [13–16]. This provides the definition of a cobordant geometry, whose connection $\bar{\omega}^{AB}$ on ∂M satisfies

$$\bar{\omega}^{1a} = 0 \,, \qquad \bar{\omega}^{ab} = \omega^{ab} \,. \tag{2.13}$$

This can be expressed as the fact that the spin connection coming from the cobordant metric possesses only 'tangential' components. Equivalently, the SFF defined in eq. (2.11) has only 'normal' components,

$$\theta^{1a} = -K^a_i dx^i \,, \qquad \theta^{ab} = 0 \,. \tag{2.14}$$

We remark that choosing a cobordant geometry that is locally a product metric on the boundary is only one possibility of a more general set of matching conditions to recover the explicit form of the SFF (2.14). We also stress that the connection $\bar{\omega}^{ab}$ needs only to be specified on the boundary ∂M, where it agrees with the dynamical field ω^{ab}. In this sense, the formalism presented here conceptually differs from any background-dependent procedure, because the latter requires the substraction of a vacuum configuration defined in the entire manifold, not just on the boundary.

From the definition of the Lorentz curvature two-form, we obtain the following decomposition for $\hat{R}^{AB}$ on the boundary

$$\hat{R}^{1a} = D_i(\omega)\theta^{1a}_j dx^i \wedge dx^j \,, \tag{2.15}$$

$$\hat{R}^{ab} = \left(\frac{1}{2}R^{ab}_{ij} + \theta^a_{i1}\theta^{1b}_j\right) dx^i \wedge dx^j \tag{2.16}$$

where $R^{ab} = \frac{1}{2} R^{ab}_{ij} dx^i \wedge dx^j$ is the boundary 2-form curvature associated to ω^{ab}. Here, we have neglected the components along dr because the boundary ∂M is defined for a fixed $r = r_0$. Using the eq. (2.14) and the projection to the basis space

$$\hat{R}^{AB}_{\mu\nu} = \hat{R}^{\lambda\rho}_{\mu\nu} e^A_\lambda e^B_\rho \tag{2.17}$$

we see that the above relations are nothing but the standard Gauss-Coddazzi decomposition of the Riemann tensor for a radial foliation (2.5)

$$\hat{R}^{rl}_{ij} = -\frac{1}{N} \nabla_{[i} K^l_{j]} \,, \tag{2.18}$$

$$\hat{R}^{kl}_{ij} = R^{kl}_{ij} - K^k_i K^l_j + K^l_i K^k_j \,. \tag{2.19}$$

Notice the change in the relative sign in eq. (2.19) with respect to a timelike (ADM) foliation.

In sum, we have chosen an adapted coordinates frame to be able to express the different components of the tetrad and the spin in terms of relevant tensorial quantities on the boundary. We know that the boundary vierbein e^a (sometimes also known as First Fundamental Form) is equivalent to the boundary metric h_{ij}, the SFF corresponds to the extrinsic curvature K_{ij} and the boundary curvature R^{ab} is the boundary Riemann tensor R^{kl}_{ij}. Then, in practice, the introduction of a reference spin connection is motivated by the need of eliminating the explicit dependence of B_3 on the components ω^{ab}, because they are part of a connection (Christoffel symbol) for the boundary metric.

2.2 Integration of the boundary term in $D = 4$

In order to have a well-defined action principle (the action to be stationary under arbitrary variations of the fields) we are going to consider again the asymptotic condition (1.3) to make the surface term (2.4) vanish. Then, assuming that the spacetime has constant negative curvature at ∂M ($\hat{R}^{AB} = -\frac{1}{l^2} e^A e^B$) and developing the surface term along the different components, we have

$$\Theta = \frac{l^2}{16\pi} \hat{\epsilon}_{1abc} \left(\delta\omega^{1a} \hat{R}^{bc} + \delta\omega^{ab} \hat{R}^{1c} \right) + \delta B_3 \,. \tag{2.20}$$

We introduced a Levi-Civita tensor for the boundary submanifold as $\hat{\epsilon}_{1abc} = -\epsilon_{abc}$, and with this notation the eq. (2.20) takes the form

$$\Theta = \frac{l^2}{16\pi} \epsilon_{abc} \left[\delta K^a \left(R^{bc} - K^b K^c \right) + \delta\omega^{ab} D K^c \right] + \delta B_3 \tag{2.21}$$

where we have use the Gauss-Coddazzi relations (2.15), (2.16). The first term can be written as

$$\epsilon_{abc} \delta K^a R^{bc} = \delta \left(\epsilon_{abc} K^a R^{bc} \right) - \epsilon_{abc} K^a \delta R^{bc} \tag{2.22}$$

where the second contribution contains the variation $\delta R^{bc} = D(\delta\omega^{bc})$. Integrating by parts (and dropping the total derivative because we are already at the boundary), we see that the term that comes out is exactly the same (with opposite sign) as the third one in eq. (2.21).

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Therefore, we are able to integrate out the boundary term B_3, demanding that the total surface term Θ vanishes

$$B_3 = \frac{l^2}{16\pi}\epsilon_{abc}K^a\left(R^{bc} - \frac{1}{3}K^b \wedge K^c\right). \tag{2.23}$$

Finally, in standard tensorial notation we write down the above expression as

$$B_3 = -\frac{l^2}{32\pi}d^3x\sqrt{-h}\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}K^{i_1}_{j_1}\left(R^{i_2i_3}_{j_2j_3} - \frac{2}{3}K^{i_2}_{j_2}K^{i_3}_{j_3}\right) \tag{2.24}$$

where $\delta^{[j_1j_2j_3]}_{[i_1i_2i_3]}$ is the completely antisymmetrized product of Kronecker deltas in the boundary indices.

It is already evident from the above expression that B_3 does not contain any term proportional to $\sqrt{-h}K$ (Gibbons-Hawking term), where $K = K_{ij}h^{ij}$ is the trace of the extrinsic curvature [17]. This is not surprising since we are dealing with an action principle different from the standard one based on a Dirichlet boundary condition for the metric.

As we shall see below, this boundary term will provide a easy-to-use, tensorial formula for the conserved charges in four dimensional AdS gravity through the direct application of Noether theorem for asymptotic symmetries.

3. Conserved quantities

3.1 General formula

Let us consider an action that is the integral of a $D-$form lagrangian density in D dimensions

$$L = \frac{1}{D!}L_{\mu_1\ldots\mu_D}dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_D}\,. \tag{3.1}$$

An arbitrary variation $\bar{\delta}$ acting on the fields can be always decomposed in a functional variation δ plus the variation due to an infinitesimal change in the coordinates $x'^{\mu} = x^{\mu} + \eta^{\mu}$. For a p-form field φ, the latter variation is given by the Lie derivative $\mathcal{L}_\eta\varphi$ along the vector η^μ , that can be written as $\mathcal{L}_\eta\varphi = (dI_\eta + I_\eta d)\varphi$, where d is the exterior derivative and I_η is the contraction operator.[1] The action of the functional variation δ on L produces the equations of motion (Euler-Lagrange) plus a surface term $\Theta(\varphi, \delta\varphi)$ and, at the same time, the Lie derivative contributes only with another surface term because $dL = 0$. Therefore, the Noether's theorem states that there exists a conserved current associated to the invariance under diffeomorphisms of the lagrangian L, that is given by [15, 18]

$$*J = -\Theta(\varphi, \delta\varphi) - I_\eta L\,. \tag{3.2}$$

For a diffeomorphism ξ that is an isometry, the total variation $\bar{\delta}$ vanishes and then all the functional variations of the fields in Θ are replaced by the corresponding Lie derivative $\delta\varphi = -\mathcal{L}_\xi\varphi$ (see also [19] and, for a recent discussion, [20, 21]).

[1]The action of the contraction operator I_η over a p -form $\alpha_p = \frac{1}{p!}\alpha_{\mu_1}\cdots_{\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$ is given by $I_\eta\alpha_p = \frac{1}{(p-1)!}\eta^\nu\alpha_{\nu\mu_1}\cdots_{\mu_{p-1}}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_{p-1}}$.

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The conservation equation for the current $d * J = 0$ expresses that $*J$ can always be written locally (by virtue of the Poincaré's lemma) as a total derivative. However, only when it can be written as an exact form $*J = dQ(\xi)$ globally, we can integrate the charge $Q(\xi)$ in a $(D-2)-$dimensional asymptotic surface (usually the boundary of the spatial section, at constant time). This is exactly what we show below for the Einstein-Hilbert action with the boundary term (2.23).

Plugging the boundary term B_3 into the expression for the total surface term (2.21), we obtain

$$\Theta = \frac{l^2}{16\pi}\epsilon_{abc}\delta K^a \left(R^{bc} - K^b K^c + \frac{e^b e^c}{l^2}\right), \tag{3.3}$$

that contains functional variations of the extrinsic curvature. In this case, the current (3.2) takes the explicit form

$$*J = \frac{l^2}{16\pi}\epsilon_{abc}\left[\mathcal{L}_\xi K^a \left(R^{bc} - K^b K^c + \frac{e^b e^c}{l^2}\right) - I_\xi D K^a \left(R^{bc} - K^b K^c + \frac{e^b e^c}{l^2}\right)\right] \tag{3.4}$$

where we have used the relation (2.15) and dropped all the components along dr. The Lie derivative on K^a can be read off from the corresponding components of the general expression for the spin connection

$$\mathcal{L}_\xi \omega^{AB} = D I_\xi \omega^{AB} + I_\xi \hat{R}^{AB} \tag{3.5}$$

that, on the boundary ∂M, is simply written as

$$\mathcal{L}_\xi K^a = D I_\xi K^a + I_\xi D K^a\,. \tag{3.6}$$

Finally, as the boundary torsion $T^a = De^a$ vanishes and the Bianchi identity for the submanifold indices reads $D(R^{bc} - K^b K^c) = 0$, we are able to write down the current as an exact form. We also assume the topology of the manifold to be $R \times \Sigma$ (with Σ as the spatial section) and that the fields fall off rapidly enough to ensure the convergence of the charge. Then, the conserved quantity associated to the asymptotic symmetry ξ is given by the integral at the boundary of the spatial section

$$Q(\xi) = \frac{l^2}{16\pi}\int_{\partial\Sigma} \epsilon_{abc} I_\xi K^a \left(R^{bc} - K^b K^c + \frac{e^b e^c}{l^2}\right). \tag{3.7}$$

In standard tensorial notation, the charge (3.7) reads

$$Q(\xi) = \frac{l^2}{32\pi}\int_{\partial\Sigma} \sqrt{-h}\,\epsilon_{i_1 i_2 i_3}\xi^k K_k^{i_1}\left(R_{mn}^{i_2 i_3} - 2K_m^{i_2}K_n^{i_3} + \frac{2}{l^2}\delta_m^{i_2}\delta_n^{i_3}\right) dx^m \wedge dx^n \tag{3.8}$$

where now all indices are spacetime ones at the boundary and $dx^m \wedge dx^n$ is the infinitesimal surface element of $\partial\Sigma$.

3.2 Charges in four-dimensional Kerr-AdS

The line element for the rotating solution in 4 dimensions can be written in Boyer-Lindquist coordinates as [22]

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - \frac{a}{\Xi}\sin^2\theta d\phi\right]^2 + \frac{\rho^2 dr^2}{\Delta} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[a dt - \frac{r^2 + a^2}{\Xi}d\phi\right]^2, \tag{3.9}$$

where the functions in the metric in terms of the spin parameter a are

$$\Delta \equiv \left(r^2+a^2\right)\left(1+\frac{r^2}{l^2}\right)-2mr\,, \tag{3.10}$$

$$\Delta_\theta \equiv 1-\frac{a^2}{l^2}\cos^2\theta\,, \tag{3.11}$$

$$\rho^2 \equiv r^2+a^2\cos^2\theta\,, \tag{3.12}$$

$$\Xi \equiv 1-\frac{a^2}{l^2}\,. \tag{3.13}$$

Kerr-AdS black hole possesses an event horizon located at the radius $r=r_+$ such that it is the largest solution of the equation $\Delta(r_+)=0$.

The formula (3.7) for the vectors $\partial_t=\partial/\partial t$ and $\partial_\phi=\partial/\partial\phi$ for a rotating black hole, evaluated on the sphere S^2 for $r\to\infty$, gives the results

$$Q\left(\partial_t\right)=\frac{m}{\Xi}\,, \tag{3.14}$$

$$Q\left(\partial_\phi\right)=\frac{ma}{\Xi^2} \tag{3.15}$$

where $Q\left(\partial_\phi\right)$ corresponds to the angular momentum J. However, the first quantity $\tilde{E}=Q\left(\partial_t\right)$ cannot be regarded as the energy for the Kerr-AdS black hole, because the Killing field ∂_t is rotating even at radial infinity. In Boyer-Lindquist coordinates, the nonrotating timelike Killing vector is the combination $\partial_t+\left(a/l^2\right)\partial_\phi$, that substituted in the charge formula (3.7) gives the *physical* energy E

$$E=Q\left(\partial_t+\frac{a}{l^2}\partial_\phi\right)=\frac{m}{\Xi^2}\,, \tag{3.16}$$

in agreement with different methods in the literature [23 – 28].[2] The relevance of this result has been emphasized by Gibbons, Perry and Pope in the context of the first law of black hole thermodynamics [26], that is not satisfied by other expressions for the Kerr-AdS energy previously found in the literature [29, 30].

4. Black hole thermodynamics in four dimensions

In this section, we use the boundary term (2.24) to cancel the divergences at radial infinity that appear in the explicit evaluation of the bulk euclidean action. For a given black hole solution, the euclidean continuation considers the horizon as shrunk to a point at the origin. The requirement that the solution be smooth at the horizon fixes the period of the euclidean time β (the inverse of the temperature T).

4.1 Kerr-AdS

We illustrate the finiteness of the euclidean action with the addition of B_3 (2.1) for Kerr-AdS black hole as a nontrivial example. In this case, the euclidean period is given by the

[2]Notice that this result differs from the energy in [3], obtained for a different choice of the Kerr-AdS metric.

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expression

$$\beta = T^{-1} = \frac{4\pi \left(r_+^2 + a^2\right)}{r_+ \left(1 + \frac{a^2}{l^2} + 3\frac{r_+^2}{l^2} - \frac{a^2}{r_+^2}\right)}. \tag{4.1}$$

The angular velocity of the black hole is

$$\Omega = \frac{a\left(1 + \frac{r_+^2}{l^2}\right)}{r_+^2 + a^2} \tag{4.2}$$

that is measured respect to a frame that is not rotating at infinity [26].

In the canonical ensemble, the euclidean action I_E is given by the free energy, $I_E = \beta F$ that satisfies the thermodynamic relation

$$E - TS - \Omega J = TI_E \tag{4.3}$$

and defines the energy E and the entropy S of a black hole for a fixed surface gravity (temperature) and angular velocity on the horizon.

The evaluation for Kerr-AdS metric of the Wick-rotated version of the action (2.1) produces the finite value

$$I_E = \frac{\pi \left(r_+^2 + a^2\right)^2 \left(1 - \frac{r_+^2}{l^2}\right)}{\Xi l^2 \left(\frac{3r_+^4}{l^2} + \left(1 + \frac{a^2}{l^2}\right) r_+^2 - a^2\right)} \tag{4.4}$$

in agreement with the standard result in the literature. The divergences at radial infinity in the bulk action are exactly canceled by the contributions from the boundary term (2.24) at $r = \infty$ (notice a sign change because of the boundary orientation). At this point, we stress that, once the temperature is fixed (in order to avoid the presence of a conical singularity), this procedure requires neither the introduction of the horizon as a new boundary nor ad-hoc boundary conditions on it, as claimed in [31].

Finally, with the expressions for the energy E (3.16) and angular momentum J (3.15) obtained in the previous section, the eq. (4.3) gives the standard result for the black hole entropy

$$S = \pi \frac{r_+^2 + a^2}{\Xi} = \frac{1}{4}\text{Area}\,. \tag{4.5}$$

5. Discussion

The standard counterterms approach considers boundary terms that are local functional of the boundary metric h_{ij} and Riemann tensor R_{ij}^{kl} and its covariant derivatives $\nabla_m R_{ij}^{kl}$, and provides a systematic method to construct them. However, in practice, for a given dimension the number of possible counterterms increase drastically as we study more complex solutions. Moreover, the extra terms needed for the convergence of the stress tensor and the euclidean action do not seem to obey any particular pattern [32]. In that spirit, one might naturally wonder if there is any other (more compact) counterterms series that also regularize the AdS action. We have shown in the previous sections that it is indeed possible to construct such series of Lorentz-covariant counterterms.

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Having the explicit form of the SFF (2.14), we can always write down the boundary term (2.23) in a fully Lorentz-covariant way

$$B_3 = \frac{l^2}{32\pi}\hat{\epsilon}_{ABCD}\theta^{AB}\left(R^{CD} + \frac{1}{3}\theta^C_F\theta^{FD}\right). \tag{5.1}$$

The reader can be convinced that any linear combination of the above terms and the expression $\hat{\epsilon}_{ABCD}\theta^{AB}e^C e^D$ (the fully-covariant version of the Gibbons-Hawking term) exhausts all possible Lorentz-covariant boundary terms for 4-dimensional gravity constructed up with the Levi-Civita as invariant tensor (and therefore, with the same parity as the bulk terms in the Einstein-Hilbert action). Logically, the term containing the tetrad does not appear in the final form of B_3, because its variation would include δe^a, what would necessarily lead us back to a Dirichlet condition for the boundary metric.

The fully-covariant expression for B_3 coincides with the boundary term present in the Euler theorem [13]

$$\int_M \hat{\epsilon}_{ABCD}\hat{R}^{AB}\hat{R}^{CD} = 32\pi^2\chi(M) + 2\int_{\partial M}\hat{\epsilon}_{ABCD}\theta^{AB}\left(R^{CD} + \frac{1}{3}\theta^C_F\theta^{FD}\right) \tag{5.2}$$

where $\chi(M)$ stands for the Euler characteristic of the manifold M. As $\chi(M)$ is a topological number (a constant), the above relation simply means that — from the dynamical point of view — a variation of the Euler term in the l.h.s. is equivalent to the variation of the boundary term B_3.

In the previous section, we used the boundary term B_3 to regularize the euclidean bulk action. However, we can also consider the regularizing effect of the Euler term in the bulk, evaluating the euclidean continuation of the action (1.2). In this case, the euclidean action appears just shifted in a constant respect to the expression (4.4)

$$I^4_E = I_E + \pi l^2 \tag{5.3}$$

that, as a consequence, produces the entropy S'

$$S' = \frac{1}{4}\text{Area} + \pi l^2\,. \tag{5.4}$$

This constant is irrelevant for the thermodynamic description of the system, but has a clear geometrical meaning. In fact, plugging the Euler-Gauss-Bonnet term from eq. (5.2) into the action (1.2) we find that both approaches are equivalent up to an *integration* constant given in terms of the Euler characteristic as $\frac{\pi l^2}{2}\chi(M)$. This feature already is present in the evaluation of the entropy for topological Schwarzschild-AdS black holes

$$ds^2 = -\Delta(r)^2 dt^2 + \frac{dr^2}{\Delta(r)^2} + r^2 d\Sigma^2_\gamma \tag{5.5}$$

with $\Delta^2 = \gamma - \frac{2G\mu}{r} + \frac{r^2}{l^2}$. These solutions posses a transversal section Σ_γ of constant curvature $\gamma = +1, 0, -1$ such that when we used the expression for the regularized action (1.2), the entropy gets an additional contribution $\pi l^2\gamma$.

For higher even-dimensional AdS gravity ($D = 2n$), a well-defined action principle was found in [6] supplementing the action in the Euler term

$$\mathcal{E}_{2n} = \hat{\epsilon}_{A_1 \dots A_{2n}} \hat{R}^{A_1 A_2} \dots \hat{R}^{A_{2n-1} A_{2n}} \tag{5.6}$$

and fixing its weight factor demanding the same boundary condition on the asymptotic curvature as in the four-dimensional case. The topological invariant $\mathcal{E}_{2n}$ again cancels the divergences coming from the bulk action, such that the regularized action can be written as

$$I_{2n} = -\frac{1}{16\pi} \int_M d^{2n}x \sqrt{-\hat{g}} \left(\hat{R} - 2\Lambda + \alpha_{2n} \delta^{[\nu_1 \dots \nu_{2n}]}_{[\mu_1 \dots \mu_{2n}]} \hat{R}^{\mu_1 \mu_2}_{\nu_1 \nu_2} \dots \hat{R}^{\mu_{2n-1} \mu_{2n}}_{\nu_{2n-1} \nu_{2n}} \right) \tag{5.7}$$

where the cosmological constant is $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$ and the coupling constant of the Euler term is

$$\alpha_{2n} = (-1)^n \frac{l^{2(n-1)}}{2^n n \left[2(n-1)\right]!} \, . \tag{5.8}$$

If we are interested in constructing explicitly the boundary term for this case, we have to consider the Einstein-Hilbert action in differential forms language

$$I_G = \frac{1}{16\pi(D-2)!} \int_M \hat{\epsilon}_{A_1 \dots A_{2n}} \left(\hat{R}^{A_1 A_2} e^{A_3} \dots e^{A_{2n}} + \frac{D-2}{Dl^2} e^{A_1} \dots e^{A_{2n}} \right) + \int_{\partial M} B_{2n-1} \, . \tag{5.9}$$

Following a procedure identical as the one shown in section 2.2, we are able to integrate out B_{2n-1} as

$$\begin{aligned} B_{2n-1} = (-1)^n \frac{l^{2(n-1)}}{8\pi(D-2)!} \int_0^1 dt \, \epsilon_{a_1 \dots a_{2n-1}} K^{a_1} \left(R^{a_2 a_3} - t^2 K^{a_2} \wedge K^{a_3} \right) \times \cdots \\ \cdots \times \left(R^{a_{2n-2} a_{2n-1}} - t^2 K^{a_{2n-2}} \wedge K^{a_{2n-1}} \right) \end{aligned} \tag{5.10}$$

where the factor $\left(R^{ab} - t^2 K^a \wedge K^b \right)$ appears $(n-1)$ times and the integration over the continuous parameter $t \in [0,1]$ gives the relative coefficients in the binomial expansion. This boundary term can be also cast in tensorial form

$$\begin{aligned} B_{2n-1} = (-1)^{n+1} \frac{l^{2(n-1)}}{2^{n+2}\pi(D-2)!} \int_0^1 dt \, d^{2n-1}x \sqrt{-h} \delta^{[j_1 \dots j_{2n-1}]}_{[i_1 \dots i_{2n-1}]} K^{i_1}_{j_1} \left(R^{i_2 i_3}_{j_2 j_3} - 2t^2 K^{i_2}_{j_2} K^{i_3}_{j_3} \right) \times \cdots \\ \cdots \times \left(R^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}} - 2t^2 K^{i_{2n-2}}_{j_{2n-2}} K^{i_{2n-1}}_{j_{2n-1}} \right) \end{aligned} \tag{5.11}$$

with the use of the totally antisymmetric Kronecker delta. The reader can notice by simple inspection of formula (5.11) that the absence of the Gibbons-Hawking term is not a particular property of the boundary term in four dimensions (2.24), but the general rule for all even dimensions in this framework.

The Noether theorem provides the expression for the conserved quantities associated to an asymptotic symmetry ξ, given by

$$\begin{aligned} Q(\xi) = (-1)^{n+1} \frac{l^{2(n-1)}}{2^{n+2}\pi(D-2)!} \int_{\partial\Sigma} \sqrt{-h} \epsilon_{i_1 \dots i_{2n-1}} \xi^k K^{i_1}_k \Bigg(\hat{R}^{i_2 i_3}_{m_1 m_2} \dots \hat{R}^{i_{2n-2} i_{2n-1}}_{m_{2n-3} m_{2n-2}} + \\ + (-1)^n \frac{2^{n-1}}{l^{2(n-1)}} \delta^{i_2}_{m_1} \dots \delta^{i_{2n-1}}_{m_{2n-2}} \Bigg) \times \\ \times d\sigma^{m_1 \dots m_{2n-2}} \, , \end{aligned} \tag{5.12}$$

where $d\sigma^{m_1 \dots m_{2n-2}} = dx^{m_1} \wedge \dots \wedge dx^{m_{2n-2}}$ is the infinitesimal surface element of $\partial\Sigma_{D-2}$.

As an example, we can compute the conserved quantities for a six-dimensional Kerr-AdS black hole with a single rotation parameter [29]

$$
\begin{aligned}
ds^2 =& -\frac{\Delta}{\rho^2}\left[dt - \frac{a}{\Xi}\sin^2\theta d\phi\right]^2 + \frac{\rho^2 dr^2}{\Delta} + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \\
&+\frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[adt - \frac{r^2+a^2}{\Xi}d\phi\right]^2 + r^2\cos^2\theta d\Omega_2^2\,,
\end{aligned}
\tag{5.13}
$$

where

$$
\Delta \equiv \left(r^2+a^2\right)\left(1+\frac{r^2}{l^2}\right) - \frac{2m}{r}\,, \tag{5.14}
$$

and $d\Omega_2^2$ is the line element on unit S^2. The other functions in the metric remain the same as in the four-dimensional case.

Evaluating eq. (5.12) for the Killing fields ∂_t and ∂_ϕin this metric, we have

$$
Q\left(\partial_t\right) = \frac{4\pi}{3}\frac{m}{\Xi}\,, \tag{5.15}
$$

$$
Q\left(\partial_\phi\right) = \frac{2\pi}{3}\frac{ma}{\Xi^2} \tag{5.16}
$$

where $Q\left(\partial_\phi\right)$ corresponds to the angular momentum J. As we had already pointed out in the four-dimensional case, the timelike Killing vector that is not rotating at infinity is $\partial_t + \left(a/l^2\right)\partial_\phi$, and produces the physical energy E

$$
E = Q\left(\partial_t + \frac{a}{l^2}\partial_\phi\right) = \frac{2\pi}{3}\frac{m}{\Xi}\left(1+\frac{1}{\Xi}\right), \tag{5.17}
$$

in agreement with the first law of black hole thermodynamics [26].

The regularized euclidean action (5.9) in six dimensions for the Kerr-AdS solution (5.13) gives

$$
I_6 = \beta\frac{\pi}{3\Xi}\left(M - \frac{r_+^2}{l^2}\left(r_+^2+a^2\right)\right) \tag{5.18}
$$

that, using the energy (5.17), the angular momentum (5.16) and the angular velocity (4.2), produces the correct value for the entropy

$$
S = \frac{2\pi^2}{3\Xi}r_+^2\left(r_+^2+a^2\right) \tag{5.19}
$$

that has also been computed by several authors using different methods [29, 32, 33, 26 – 28].

Finally, in $D = 2n$ we can also write down the fully Lorentz-covariant version of the boundary term (5.10) and discover the connection with the Euler term (5.6) through the Euler theorem in higher even dimensions. As a consequence, again both procedures to compute the euclidean action (with the boundary term B_{2n-1} or with $\mathcal{E}_{2n}$ as a bulk term) are simply related by a topological number.

6. Conclusions

In this paper, we have explicitly constructed the Lorentz-covariant counterterms that regularize the action for AdS gravity in four dimensions. Our starting point was a well-defined action principle consistent with a boundary condition for the asymptotic curvature. Certain choice of the orthonormal frame allows us to write down the boundary term and the Noether charges associated to asymptotic symmetries in standard tensorial formalism.

We have also explored the connection between these regularizing counterterms and topological invariants. As the Euler term is dynamically equivalent to a boundary term (by virtue of the Euler theorem), the divergences in the euclidean action can be equally canceled by the bulk or the surface term.

It is remarkable how a single boundary condition achieves a finite action principle: the action is stationary under arbitrary variations of the fields and the conserved charges and the euclidean action are finite. On the contrary to an action principle that relies on a Dirichlet condition for the metric, where we can add any boundary term that is a functional of the boundary metric, here we can only incorporate boundary terms whose variations are compatible with the asymptotic condition (1.3). This simply means that even though we can always add surface terms to the action, in general, the addition of an arbitrary boundary term will spoil the boundary condition. This argument seems to explain why, in the end, we have a quite restrictive action principle in spite of a general assumption for the boundary condition.

More technically, the fact that the boundary term B_3 is constructed using totally antisymmetric $3-$forms seems to rule out many of the terms present in the standard counterterms series. For instance, the present formalism cannot include any term that contains covariant derivatives of the intrinsic curvature $\nabla_m R_{ij}^{kl}$ because they would be automatically eliminated by the Bianchi identity.

In this paper, we have just given one explicit example in higher even dimensions. However, it is expected that the same arguments about the convergence of the euclidean action and the conserved quantities hold for any dimension $D = 2n$. We hope to report this elsewhere.

Although the boundary term B_3 substantially differs in its form from the standard counterterms in four dimensions [32], it is clear that both approaches cancel the same divergent powers in r arising from the bulk action. A comparison might be performed for a particular coordinates choice, for instance, using the Fefferman-Graham expansion for the metric [34], suitable to describe the conformal structure of an asymptotically AdS spacetime

$$ds^2 = \frac{l^2}{4\rho^2} d\rho^2 + \frac{g_{ij}(\rho, x)}{\rho} dx^i dx^j \tag{6.1}$$

where the boundary is located at $\rho = 0$ and

$$g_{ij}(\rho, x) = \overset{(0)}{g}_{ij}(x) + \rho \overset{(1)}{g}_{ij}(x) + \rho^2 \overset{(2)}{g}_{ij}(x) + \cdots \tag{6.2}$$

and $\overset{(0)}{g}_{ij}$ is a given boundary data for the metric. Indeed, a simple computation tells us that the expansion of the relevant components of the Riemann tensor (2.19) reads

$$\hat{R}^{kl}_{ij} = -\frac{1}{l^2}\delta^{[kl]}_{[ij]} + \rho\left(\overset{(0)}{R}{}^{kl}_{ij} + \frac{1}{l^2}\left(\delta^k_{[i}\,\overset{(1)}{g}{}^l_{\ j]} + \overset{(1)}{g}{}^k_{\ [i}\,\delta^l_{j]}\right)\right) + \cdots \tag{6.3}$$

where the extra terms are increasing powers of ρ and the indices of the tensorial coefficients are raised and lowered with $\overset{(0)}{g}_{ij}$. This reflects how the boundary condition (1.3), required to attain a finite action principle, is automatically satisfied in the coordinates frame (6.1).

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